Period doubling and chaos in non-linear maps

Alexander Atureliya, Aneesh Mondal, Rohan Persand, Yendo Tang
Imperial College London Mathematics School
Research Project Presentation Day, Wednesday 27 March 2024

Introduction

Systems are everywhere in the real world and modelling such systems accurately is a main goal of the mathematical sciences. However, as it turns out, a minute change in initial conditions of a system may lead to drastically different behaviour to what is expected. This is known as chaos. Suppose the atmosphere contained slightly less oxygen [1] , or the orbit of a planet was slightly closer to the sun [2]. These perturbations can interact, building upon each other which can eventually cause a completely different outcome, such as changing the habitability of the planet and causing the creation or destruction of different life forms.

The Logistic Map

Attempting to model a population

Say we wanted to model a population against time, one way we could do this would be with a basic equation such as [3]:

$$x_{n+1} = kx_n(1 - x_n)$$

where:

- x_n is the population this year.
- x_{n+1} is the population next year.
- k is the reproductive rate.
- $(1 - x_n)$ to represent a theoretical maximum population that would represent real life constraints (e.g. deaths).

Here the x_n value is taken to be a proportion of the theoretical maximum (i.e. a value between 0 and 1). This equation is called the logistic map [4]. The parabolic nature of the map means it may be regarded as a non-linear map.

Variations in k

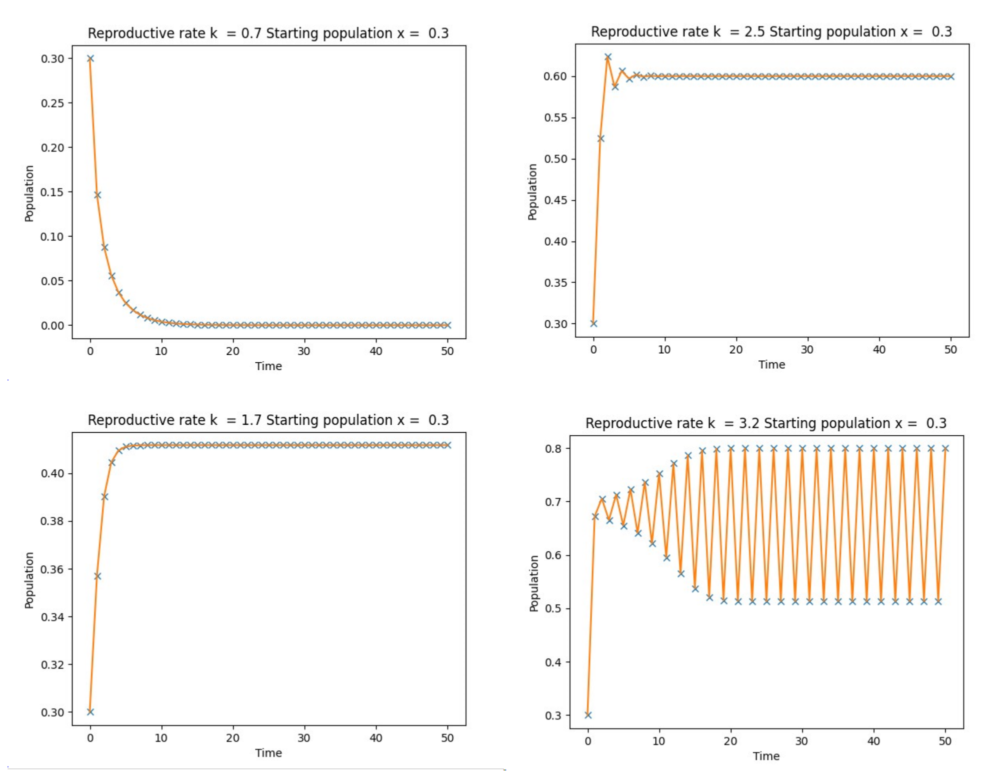

Figure 1: The populations varying with different k values, with $x_0 = 0.3$.

Figure 1 shows how a change in the reproductive rate k affects the population of our model. In the first 3 cases in figure 1, the final population eventually settles on a particular value m: when $k < 1$, the final population settles at 0, whilst when $1 < k < 2$, the population rises before settling on to a population of ≈ 0.41176. When $2 < k < 3$, for example $k = 2.5$, the initial populations follow a series of small oscillations decreasing in amplitude eventually settling onto a value of ≈ 0.6000. When $k > 3$ ($k = 3.2$ in this case), the final population oscillates between 2 different values, for the moment.

The bifurcation diagram

Introducing the bifurcation diagram

As it is the final population(s) which we are interested in, we may observe how the final population(s) vary by changing k values. We can observe the general pattern of a final population(s) with regards to a specific k value using a bifurcation diagram.

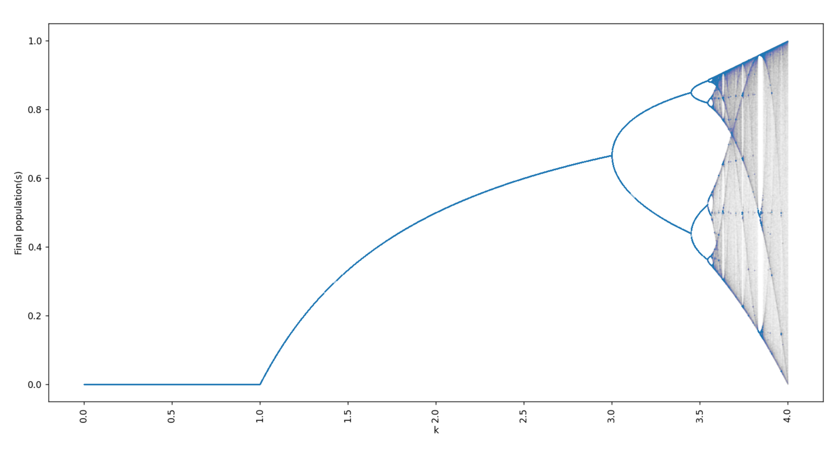

Figure 2: Bifurcation diagram for $0 \leq k \leq 4$

The y-axis represents the final population (or populations if the value oscillates), whilst the x-axis represents the reproduction rate, k, of our model. Producing such diagrams for different k leads to the following observations.

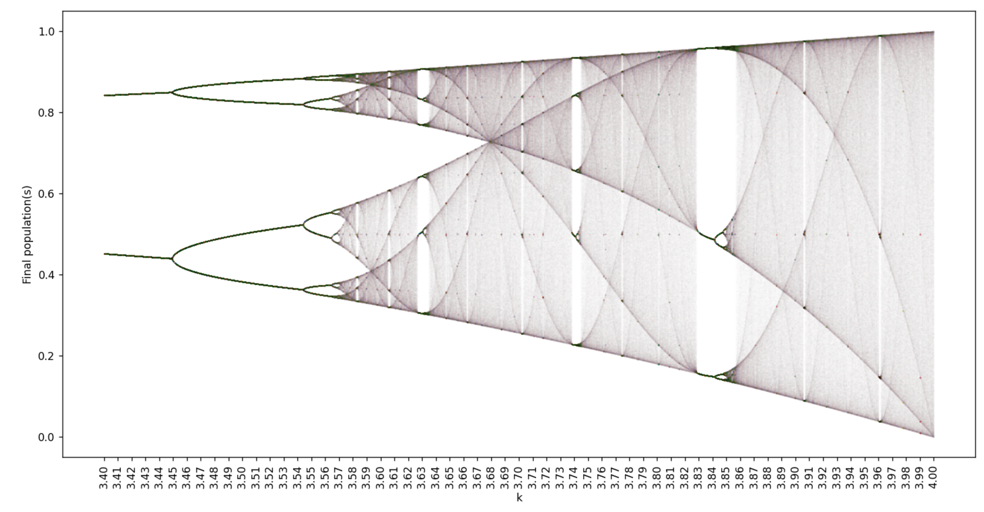

Figure 3: Bifurcation diagram for $3.4 \leq k \leq 4$

Observations

Analysing the images generated in figure 2 and 3, the following observations may be made.

Where $0 \leq k \leq 1$, the final population would be 0. Increasing the k value to be $1 < k < 3$ would result in one final fixed population. Subsequent k values follow period 2 behaviour (the final population oscillates between 2 consistent values). By the time $k = 3.45$, period 4 behaviour is demonstrated , with the the period doubling to 8 at $k = 3.544$ and period 16 behaviour at $k = 3.5644$. At $k \approx 3.57$, chaos occurs: the final populations have no distinct period. Other period behaviours are observed in the logistic map: period 3 behaviour is observed where $k = 3.83$.

Chaos

Sensitive dependence on initial conditions

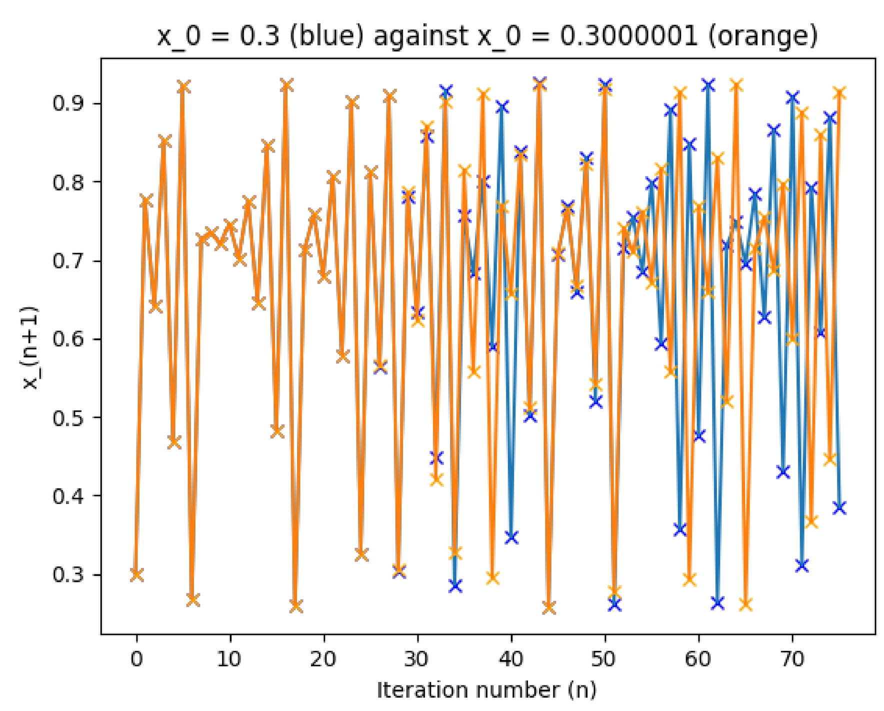

Figure 4: Visualisation of how changing $x_0 = 0.3$ to $x_0 = 0.3000001$ can affect the subsequent x_{n+1} values significantly

Figure 4 shows how the logistic map has a sensitive dependence on the initial conditions, as demonstrated by changing $x_n = 0.3$ to $x_n = 0.3000001$. After $n = 30$, the behaviour of each function begins to vary significantly; thus demonstrating how a slight change in initial conditions may produce vastly different results. In essence, this is the definition of chaos.

Cobweb plots

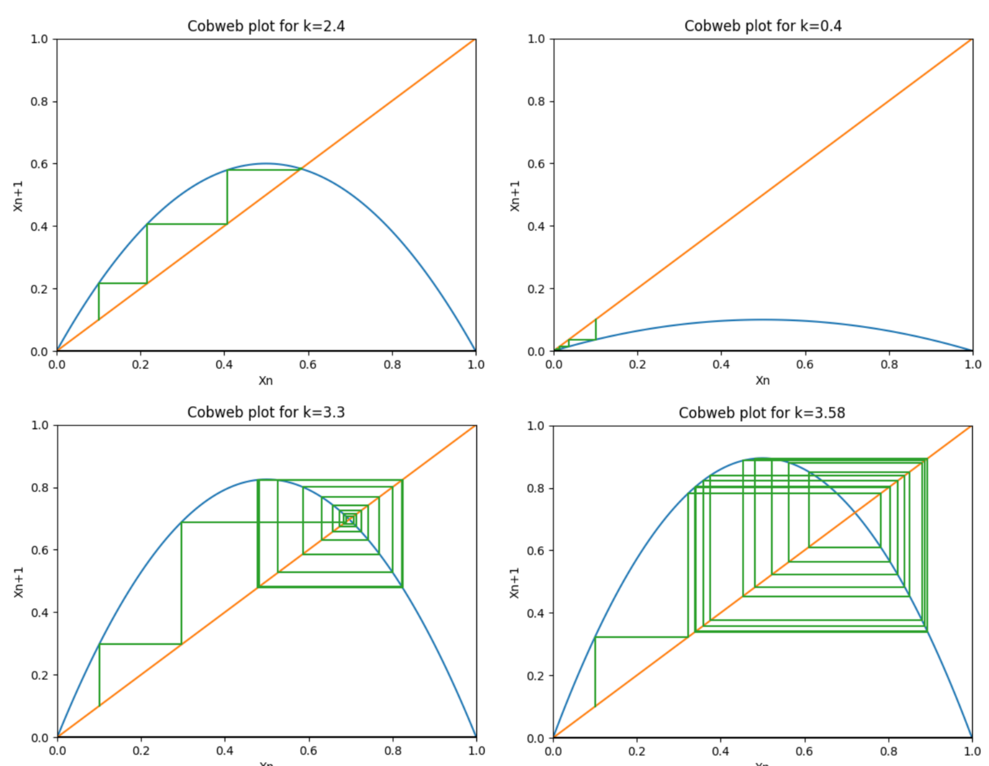

Figure 5: Cobweb plots for varying k values

Plotting x_{n+1} against x_n would give us an inverted parabola where k affects the parabolic shape: points on the parabola that intersect the line $y = x$ are called fixed points. This may be used to produce cobweb plots as shown in figure 5. The green line represents the behaviour of subsequent populations. In a nutshell, a reason for such chaotic behaviour may be determined with regards to the gradient of the map at the fixed point.

Quantifying chaos: the Lyapunov exponent

We may quantify chaos using the Lyapunov exponent: the mean of the logarithm of the absolute value of the gradient over an infinite cycle.

$$\ln |(f^n)'(x_1)| = \sum_{j=1}^{n} \ln |f'(x_j)|$$

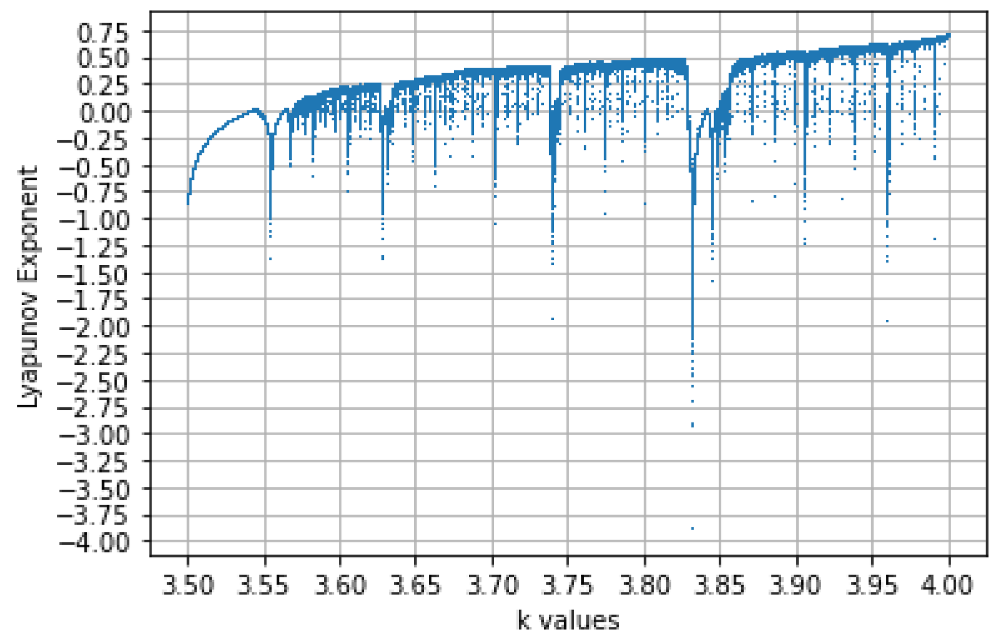

Figure 6: The Lyapunov exponent from 3.5 to 4 for 1100 iterations, retaining the last 1000

Comparing figure 6 to figure 3, where the Lyapunov exponent is greater than 0 corresponds to chaotic behaviour, below 0 corresponds to stability, and at 0 corresponds to bifurcations occurring.

Examples of chaotic

Solar System

Our current mathematical models of the solar system [2] mean we need to know the current state to an infinite precision (which is impossible), hence we do not know what the state of the Solar System will be in a billion years. Due to the chaotic nature, the error in our measurement of the current state compared to the true value will grow exponentially overtime, rendering prediction or accurate simulation useless.

Concorde

Concorde's wing design demonstrates where chaos may be useful. One of the biggest engineering challenges of Concorde was designing wings to allow the aircraft to fly at a range of speeds, from slow flight for take off and landing, to cruising at twice the speed of sound. The solution was to use chaotic turbulent air flow caused by slow moving vorticies which were shed by the large, seamless wing shape, elevons on the wing, and the steep angle of attack; all allowing for the production of lift at low airspeeds.

Conclusions

We conclude that there are severe limitations to mathematical models. Even in the simplest deterministic nonlinear model, an iterative discrete quadratic, chaotic behaviour exists. The key idea is that unpredictability does not exist because of randomness or our lack of knowledge, but because systems are inherently chaotic as a fundamental property.

References

[1] Anna Trevisan and Luigi Palatella. Chaos and weather forecasting: the role of the unstable subspace in predictability and state estimation problems. *International Journal of Bifurcation and Chaos*, 21(12):3389–3415, 2011.

[2] Jacques Laskar. Large-scale chaos in the solar system. *Astronomy and Astrophysics (ISSN 0004-6361), vol. 287, no. 1, p. L9-L12*, 287:L9–L12, 1994.

[3] Steven H Strogatz. *Nonlinear dynamics and chaos with student solutions manual: With applications to physics, biology, chemistry, and engineering*. CRC press, 2018.

[4] Veritasium. This equation will change how you see the world (the logistic map). Video, 2020. URL `https://youtu.be/ovJcsL7vyrk?si=5P5EkznGDyqSfjAs`. Accessed 28th January 2024.